Constructing a Square with the Area 1/n of a Given Square

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We present here a solution to the problem posed on page **26**

A construction to halve the area of a given square.

In the following figure let *ABCD* be a given square (of unknown side length). Let *PB* = *BS* = 1 unit, and *BQ* = 2 units. Let *PRQ* be a semicircle passing through points *P* and *Q*. Then the area of the square *XYZB* must be half of that of *ABCD*.

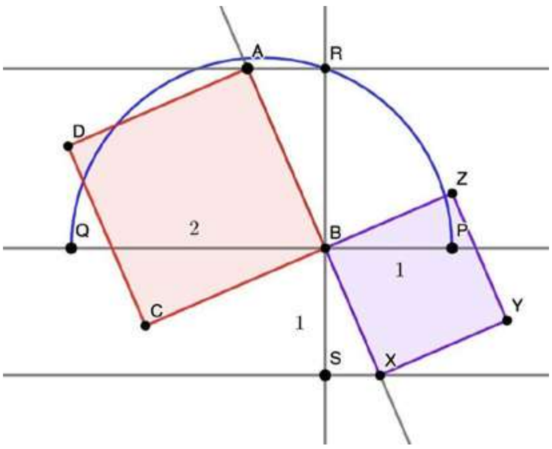

Figure 1

This question and this picture were submitted by one of our authors and it made us think. Do spend some time observing this picture – Is the area of square *XYZB* half the area of the given square *ABCD*? If so, why? We explain below.

If we apply computational thinking to solve this problem, this is the first step:

Statement of the Problem: Is the area of square *XYZB* half the area of the given square *ABCD*? If so, why?

Let us try to decompose the steps: We are given that *QB* = 2 units, and *PB* = *BS* = 1 unit.

Given a square whose area is x square units, we want to draw a square whose area is $\frac{x}{2}$ square units. A quick observation reveals that this is the same as given a line segment of $\sqrt{x}$ units, we want to construct a line segment of $\sqrt{\frac{x}{2}}$ units.

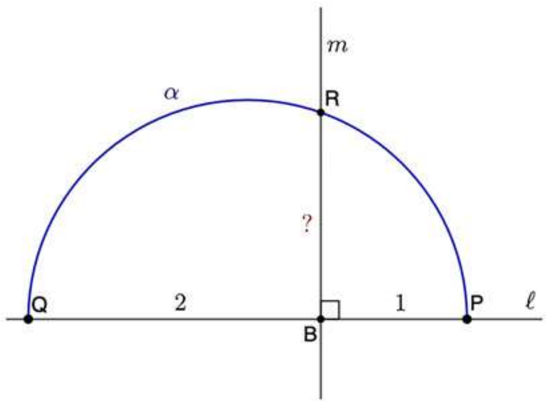

Figure 2

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Step 1. Mark three points P, B and Q on a line l, such that B lies in between P and Q. $QB = 2$ units and $PB = 1$ unit.

Step 2. We draw a semicircle α with PQ as the diameter.

Step 3. Let the line m perpendicular to l and passing through B intersect α at R. (See Figure 2)

What will BR be?

From the right-angled ΔPBR, $PR^2 = PB^2 + RB^2$.

From the right-angled ΔRBQ, $QR^2 = RB^2 + BQ^2$.

From the right-angled ΔPRQ, $PQ^2 = PR^2 + RQ^2$ (note that $\angle PRQ$ is a right angle in the semicircle).

By using these three equations, we arrive at the fact that $BR = \sqrt{2}$.

Step 4. Mark the point S on the line m at the other side of l such that $BS = 1$ unit (see Figure 3)

Step 5. Draw the line α parallel to l passing through R, and the line b parallel to l passing through S.

Step 6. Choose any point A on the line α and join the points B and A via the line c. Let c intersect b at X.

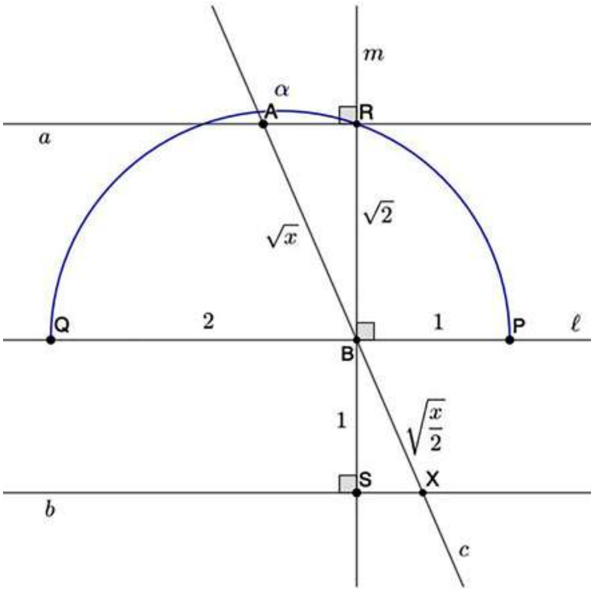

Figure 3

Now ΔARB and ΔXBS are similar. So, if $AB = \sqrt{x}$ units, then $BX = \sqrt{\frac{x}{2}}$ units.

Thus, if we construct a square with AB and XB as respective side lengths, then necessarily we should have that the square with AB as a side, must be two times the area of the square with XB as a side. Thus, by decomposing the solution steps, we conclude that the area of square $XYZB$ is indeed half the area of the given square $ABCD$.

The reader might have already observed that there are simpler constructions of squares that can halve the area of a given square, such as in Figure 4. Here $ABCD$ is a given square, and P, Q, R and S are midpoints of sides AB, BC, CD and DA respectively. However, the above construction can be extended to construct a square with area $\frac{1}{n}$ of a given square, by taking $BQ = n$ instead of 2 (See Problem 1 below).

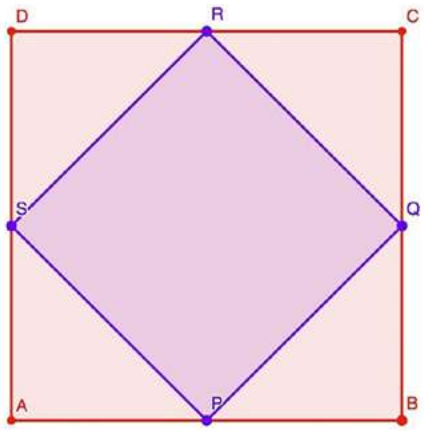

Figure 4

Why stop at halving the area of a square? We claim that this construction can do more! We provide a set of problems for the reader to try their hands on.

Problems

1. If we assume that $BQ = n$ units, what would be the length of BR? What would be the length of BX? What is the ratio between the areas of $ABCD$ and $XYZB$?
2. The above construction works only if the side AB is more than or equal to BR. What if $AB<BR$? The above construction can be

slightly modified to accommodate this possibility (Figure 5). Argue why the following construction would work. Here β is a semicircle joining the points B and R, and γ is the semicircle joining the points B and S.

3. Do the above constructions work if we want to halve the area of an equilateral triangle? A regular hexagon? A regular 13-gon? A circle?
4. Given a two dimensional shape, can you argue if the above constructions give a shape which is similar to the given shape and has area $1/n$ of that of the given shape?

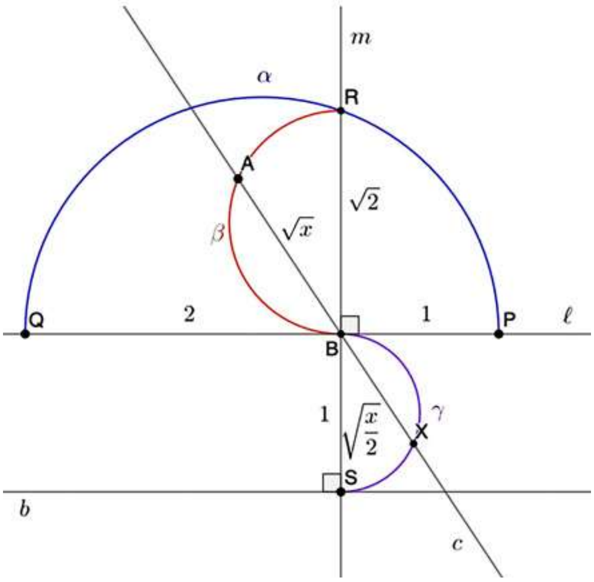

Figure 5

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