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ClassRoom

Explorations on Triangles with Integer Sides

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In the July 2018 issue of At Right Angles, author A S Rajagopalan had explored the number of triangles with integer sides. He had shown that if two of the sides are the integers a and b, with $a > b$, then the third side c can take only integer values from $a + b - 1$ to $a - b + 1$, leading to the conclusion that there can only be $2b - 1$ such triangles.

In this article, student Shreyan develops a formula for the number of triangles (a, b, c) with integer sides when b is fixed and $a \le b \le c$.

Theorem. *Let b be a fixed positive integer. Then the number of integer sided triangles ABC with sides (a, b, c) where $a \le b \le c$ is equal to $b(b + 1)/2$.*

Proof. We are going to represent the problem graphically on the coordinate plane, so it is convenient to write x for a and y for c. The task we have before us is therefore the following: to count the number of integer pairs (x, y) satisfying the following inequalities:

$$1 \le x \le b \le y,$$
$$x + b > y.$$

Consider the second inequality, $x + b > y$. As x, b, y are all integers, this inequality is equivalent to $x + b \ge y + 1$, which may be written in the form

$$y - x \le b - 1.$$

Hence the above system of inequalities may be rewritten as follows:

$$1 \le x \le b \le y,$$
$$y - x \le b - 1.$$

We need to count the number of integer pairs (x, y) satisfying these conditions. We shall do so by graphing the inequalities and counting the lattice points (i.e., the points with integer coordinates) within the feasible region. Once we draw the relevant graph, the formula for the total number of lattice points becomes transparently clear.

Figure 1 shows the four constraints plotted on a graph. For illustrative purposes, we have used the value $b = 5$.

Keywords: Integers, triangle inequality, counting, constraints, linear programming

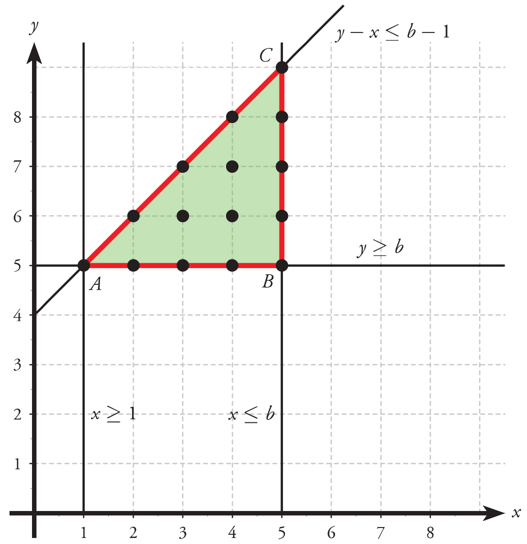

Figure 1.

Observe that the feasible region is an isosceles right-angled triangle with its vertices at $A\,(1, b)$, $B\,(b, b)$ and $C\,(b, 2b-1)$. Observe also that the lattice points within the triangle are neatly laid out in the form of a triangular array, with b lattice points on the base AB, $b-1$ lattice points one unit above the base, $b-2$ lattice points two units above the base, and so on, culminating in a single lattice point at the vertex C. Hence the total number of lattice points within the feasible region is the sum of the following arithmetic progression:

$$b + (b-1) + (b-2) + \cdots + 1,$$

which equals $\frac{1}{2}b(b+1)$, as claimed.

Non-graphical proof. If we wish to avoid arguing with reference to a graph, we could do so as follows. The system of inequalities,

$$1 \le x \le b \le y,$$
$$y - x \le b - 1,$$

may be rewritten as a single extended inequality:

$$y - b + 1 \le x \le b \le y.$$

Since $y < x + b$ and $x \le b$, we obtain $y < 2b$, i.e., $y \le 2b - 1$. Hence there is no harm in further rewriting the above extended inequality in the following form.

$$y - b + 1 \le x \le b \le y \le 2b - 1.$$

We need to count the number of integer pairs (x, y) satisfying these conditions.

- The extended inequality tells us that y can take values from b to $2b-1$ (both endpoints included).
- Suppose $y = b$; then we get $1 \le x \le b$, which means that x can take b possible integer values.
- Next, suppose $y = b + 1$; and we get $2 \le x \le b$, which means that x can take $b - 1$ possible integer values.
- Similarly, if $y = b + 2$, we find that x can take $b - 2$ possible integer values.
- This progression continues till the case $y = 2b - 1$, when x can take just 1 possible integer value.

Hence the total number of possibilities is

$$b + (b-1) + (b-2) + \cdots + 1 = \frac{b(b+1)}{2},$$

as earlier.

References

1. A S Rajagopalan, "Triangle inequality — a curious counting result" from *At Right Angles*, http://teachersofindia.org/en/article/triangle-inequality-curious-counting-result

SHREYAN JHA is an 11-year old boy who is being homeschooled. He is an avid reader who finds the world around him intriguing. He loves numbers and science and enjoys conversations about these subjects. He tends to get lost in his thoughts, and his writing is a product of such introspection. Shreyan may be reached at shivajha@gmail.com.