Indian Mathematics

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1 Introduction

This article discusses some of the significant contributions to geometry made by our ancient Indian mathematicians Brahmagupta and Narayanapandita.

2 Pythagorean (or) Brahmaguptan triplets

Let us start with the basics: the Pythagorean triplets. To summarise, a Pythagorean triple is composed of three positive numbers a, b and c.

$$a^2 + b^2 = c^2$$

In class 8, we studied an algorithm for generating triplets. Take an arbitrary number, '$m > 0$' and pass it through the following generating functions:

$$2m, \quad m^2 - 1, \quad m^2 + 1 \tag{1}$$

This, however, only produces some conceivable triplets. To generate more, replace 1 in Equation (1) with n^2 to get: $2mn, m^2 - n^2$ and $m^2 + n^2$. This general form was provided by Brahmagupta in his book [1]. Brahmaguptan (or) Pythagorean triplets find applications in (a) solving problems in trigonometry; (b) finding distances through triangles and (c) to derive a method to solve *quadratic indeterminate equations*.

3 Tetrads

Let us move on to Tetrads. Tetrads are any four positive integers that satisfy the condition

$$a^2 + b^2 = c^2 + d^2 \tag{2}$$

$a^2 + b^2 = c^2 + d^2$. What are the practical applications of this? Surprisingly, no one knows. Brahmagupta was one of the first Indian mathematicians (that we know of!) to focus on issues that were not commonly encountered. Brahmagupta just did mathematics for the fun of it. Tetrads are useful for constructing *Cyclic quadrilaterals*.

4 Cyclic Quadrilaterals

A cyclic quadrilateral is a quadrilateral whose 4 points lie on a circle. The sum of opposite angles of a cyclic quadrilateral is 180°. You may be curious how cyclic quadrilaterals evolved from Pythagorean triplets and tetrads. Well, here is the connection: the elements of a tetrad, namely a, b, c and d, will be employed to construct the sides of a cyclic quadrilateral.

Consider the equation of a tetrad, Equation (2). When both the LHS and RHS are equated to an arbitrary number, say k^2, we get:

$$a^2 + b^2 = k^2 \tag{3a}$$

$$c^2 + d^2 = k^2 \tag{3b}$$

From Equation (3), it is observed that we get two right-angled triangles with a common hypotenuse e and the opposite angles are 90° each. They sum up to 180°. Therefore, a quadrilateral with a side length of tetrads can be cyclic. Figure 1 shows a typical cyclic quadrilateral. Next, given any four sides, say a, b, c and d, 3 different cyclic quadrilaterals can be constructed/generated. Also, there are 3 different diagonals say X, Y and Z. The existence of the third diagonal was given by Narayanapandita in his book [2] Figure 1 shows two diagonals X and Y, the third diagonal Z can be obtained by flipping the two sides, c and d. On the contrary, "what will happen if we flip any other two sides?" Try it out.

Formula for the diagonal Okay, that's cool, but you must be wondering how to determine the diagonal's length. The Pythagoras theorem can be applied to diagonal X, and we can state that

$$X = \sqrt{a^2 + b^2} \quad \text{or} \quad X = \sqrt{c^2 + d^2} \tag{4}$$

Hmm.. that's easy. but what about Y though? Since there are no right angles, it's not going to be easy.

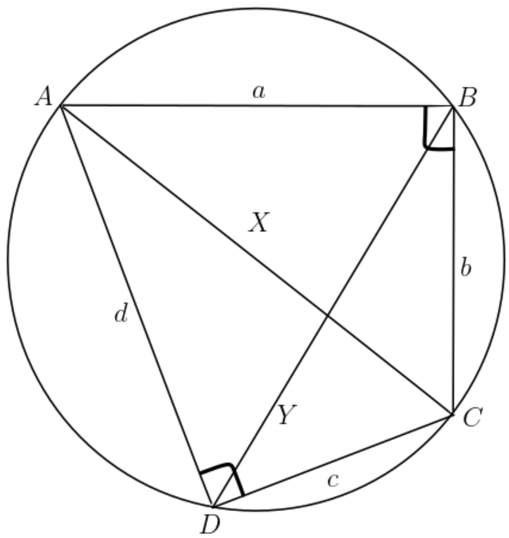

Figure 1: Schematic representation of a cyclic quadrilateral

Fortunately for us, Brahmagupta devised the following formula that we can use:

$$X = \sqrt{\frac{(ad+bc)(ac+bd)}{(ab+cd)}} \tag{5}$$

The formula to compute the lengths of the diagonals Y and Z are:

$$Y = \sqrt{\frac{(ab+cd)(ac+bd)}{(ad+bc)}} \tag{6a}$$

$$Z = \sqrt{\frac{(ab+cd)(ad+bc)}{(ac+bd)}} \tag{6b}$$

The well-known Ptolemy's theorem is obtained by multiplying the diagonals X and Y. Mathematical calculations reveal that

$$XY = ac + bd \tag{7}$$

5 Conclusion

The major conclusions from this article are:

1. that generally we can construct 3 unique cyclic quadrilaterals with any four sides. The diagonals of the constructed cyclic quadrilaterals are:
   - X, Y
   - Y, Z
   - X, Z
2. the equation to compute the length of the diagonals
3. *and most significantly*, all of this was derived from our really well-known **Pythagoras theorem!!** We seriously undermine the power of the seemingly innocent Pythagoras theorem!! In fact, it is evident everywhere!!

   And last, but not the least, we can obtain the formula for the circum-radius and the area of a cyclic quadrilateral which is derived from tetrads... or has 2 opposite right angles. If you find out how, kindly let me know.... ☺

——————☺**THANK YOU**☺——————

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References

[1] Brahmagupta's Brāhmasphuṭasiddhānta VOL I., Brahmasphutasiddhanta.

[2] Narayanapandita's Ganita-kaumudi Chap-4. Ganita-Kaumudi.