Linear Programming Problems

Vaibhav S

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1 Introduction

Linear programming was discovered during the second world war to maximise profit while using minimum resources. It used algorithms fed to computers to perform the job. However, we need to learn how to solve a problem in order to create an algorithm. This method of problem-solving only works for linear equations. The prerequisite knowledge for learning the basics is as follows:

- Solving a system of equations
- Plotting equations, inequalities, and points on the Cartesian system
- Elementary row transformations (ERT) and the basics of matrices

Before we start, it is important to know how to plot an inequality. An inequality is an expression with a greater than (>) or lesser than (<) sign. There are two other signs, namely greater than or equal to (≥) and less than or equal to (≤).
Here are the steps to plot an inequality expression on a graph:

1. Construct a line considering the inequality as an equation. For example, if we have $x + y < 4$, the line for the equation $x + y = 4$ should be constructed.
2. Then, depending on whether the inequality has an equal sign or not, we draw the line as a continuous line (just a line) or a dotted line, respectively.
   - For example, if the equation $x + y < 4$ is constructed, a dotted line will be used.
   - However, if $x + y \leq 4$ is considered, then the line will not be dotted.
   - The dotted line is to show that no value on that line will be considered, as x + y will not have solutions on that line. All solutions will be lesser than the solutions on the line.
3. If the equation contains a "less than" inequality, say $x + y < 4$, then the region below (the region where the values of x and y will be less than 4) will be shaded.
4. On the contrary, if the equation contains a greater than inequality, say $x + y > 4$, then the opposite region, the region above will be shaded.

2 A sample problem

ABC Ltd. is a company that manufactures high-quality glass products, including windows and glass doors. The company has three functioning plants: Plant 1, Plant 2, and Plant 3. Plant 1 manufactures aluminium frames and hardware, Plant 2 deals with wood frames, while Plant 3 installs the glass inside the frames and does the final assembly of the product.

The company plans to introduce two new products, ASTRA and LEO, to replace older non-profitable products. Plant 1 and Plant 3 are needed for ASTRA, requiring 1 hour and 3 hours, respectively. Meanwhile, LEO will be constructed in Plants 2 and 3, using 2 hours each in both plants.

The available production time in Plants 1, 2, and 3 are 4, 12, and 18 hours, respectively. The profit from each batch of ASTRA and LEO will be ₹3 lakhs and ₹5 lakhs, respectively. The problem is to find the most profitable product mix of ASTRA and LEO.

3 The start of the solution

The question may seem complex, but with a few good trial and error attempts, the problem can be easily solved. The following are few approaches to solve the problem:

- Graphical method
- Simplex + graphical method
- Simplex table method

For all of these methods, a table needs to be constructed, and the equations/inequalities have to be written to make the journey smoother.

	Production time (in hrs)		
Plant	ASTRA	LEO	Available production time per week
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch (in rupees)(in Lakh)	3	5	

Now, let's define the variables and write the equations: Let 'x_1' be the number of batches of ASTRA released per week, 'x_2' denote the number of batches of LEO released per week and profit in Lakhs is represented by z. Then, we have, $z = 3x_1 + 5x_2$

Objective – Maximise 'z' subject to the following constraints:

$$\begin{aligned} &x_1 \leq 4 \\ &2x_2 \leq 12 \implies x_2 \leq 6 \\ &3x_1 + 2x_2 \leq 18 \end{aligned} \tag{1}$$

These constraints are known as functional constraints. In addition, all the variables (except z) cannot be less than 0, since they represent the number of batches, and the combination of them represents time. This leads us to non-negativity constraints, given by:

$$
\begin{aligned}
&x_1 \geq 0 \\
&x_2 \geq 0 \\
&3x_1 + 2x_2 \geq 18
\end{aligned} \tag{2}
$$

3.1 The graphical method

For this method, a graph is plotted with all the given constraints/inequalities (see Figure 1). The region inside the black lines is known as the feasible region, where all the values of x and y are within the set constraints. To maximize z, a point in this region (which also includes the black lines) should be found so that when the x_1 and x_2 coordinates are plugged into the equation for z, the value of z will be maximum.

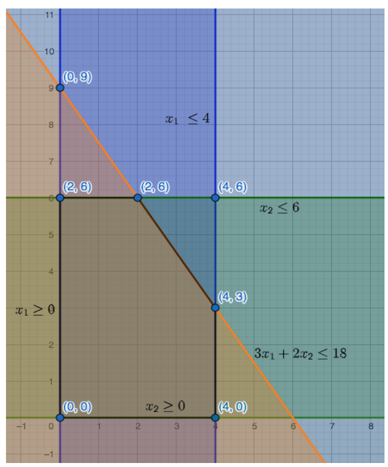

Figure 1: Graphical representation of the equations

In this method, a line is drawn for the equation $z = 3x_1 + 5x_2 = k$ where k can be any value, such as 30. It is better to draw this on a sheet of paper. It is also known that in a set of equations, if all the coefficients of the respective variables are equal and the constants are different, all those equations when plotted are parallel.

After plotting, $z = 3x_1 + 5x_2 = 30$ the scale is slowly moved parallel to the original line. The scale should be moved until we reach a point where the feasible region ends. In Figure. 2, the line intersects the point (2, 6) before going off into the non-feasible region. To find the maximum value, the coordinates of x_1 and x_2 are substituted into the 'z' equation:

$$z = 3x_1 + 5x_2 = 3(2) + 5(6) = 6 + 30 = 36 \tag{3}$$

Therefore, the final answer is ₹36 lakhs. This answer can be compared to the answer obtained from the trial and error method.

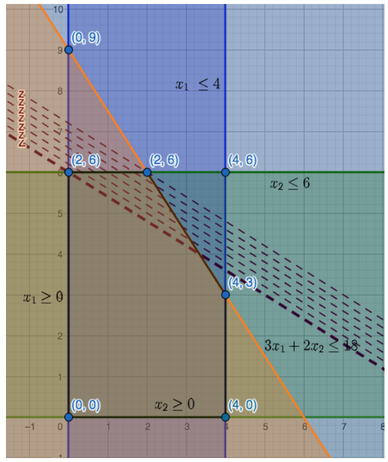

Figure 2: Graphical representation of the equations with the final profit line

3.2 The simplex - graphical method

The graphical method may not be easy to perform, especially when taking the scale and moving in parallel. It can be a confusing task and may result in errors. The next method, called the "simplex graphical method," is relatively simpler to use. In this method, the corner points will be tested instead of constructing and drawing parallel lines. First, we define two terms:-
CPF *(Corner Point Feasible solution)* = One of the corner point of the feasible region.

Optimality test = If a solution has no adjacent CPF solutions that are better, then it must be the optimal(best) solution. (When the x_1 and x_2 values of the CPF are substituted in the profit equation, is the value greater than that of the adjacent values?)

Initialisation = Set up to start iteration (with finding of a CPF solution). Start with the least value of x_1 and x_2 preferably (0,0). The steps are illustrated in Figure. 3.

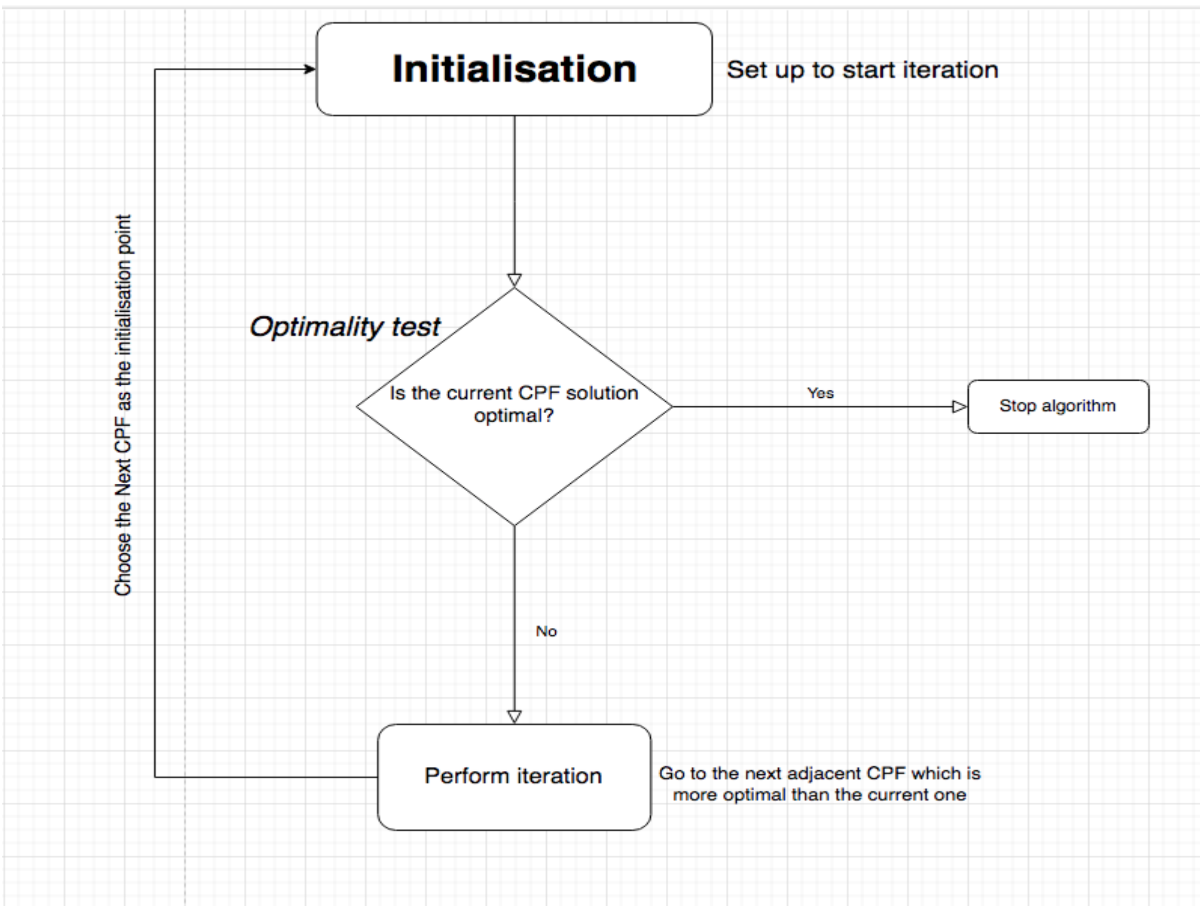

Figure 3: Flow chart for LPP Simplex algorithm

Basically, this is what happens. This method can be tried out in the sample problem given.

1. Let's start with the value that will give minimum profit, (0,0).
2. Substituting in the profit equation, the answer $z = 0$ is obtained.
3. Next, we have to check whether the 2 adjacent points yield a bigger profit.
   - Checking for point (0,6): $z = 3(0) + 5(6) =$ ₹30 lakh.
   - Checking for point (4,0): $z = 3(4) + 5(0) =$ ₹12 lakh.
   - We have found two better solutions for z. Therefore, the next iteration is performed.
4. For the next iteration, the point which yields a higher profit should be chosen.
   Since ₹30 lakh $>$ ₹12 lakh, point (0,6) should be chosen for the next iteration.
5. Again the process is repeated. But here, the point (0,0) need not be checked.
   - The adjacent CPF's are (0,0) and (2,6).
   - The point (0,9) can be ignored as it is not in the feasible region.
   - As said before, (0,0) need not be checked.
     Checking for point (2,6): $z = 3(2) + 5(6) =$ ₹36 lakh.
   - Since 36 is greater than 30, the point (2,6) is to be taken as the next iteration.
6. Again, the point (4,6) need not be checked.
   - Checking for point (4,3): $z = 3(4) + 5(3) =$ ₹27 lakh.
   - The profit value for (4,3) is lesser than that for (2,6). Therefore, the algorithm is stopped as the optimal solution is reached.
7. $\therefore z_{max} =$ ₹36 lakh.

This coincides with the answer we found earlier. This is how the Simplex-graphical method is to be done.

4 Matrices

Before the next part, the simplex table method is dealt with, an introduction to matrices and the Gauss-Jordan method is needed.

A matrix is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, treated as a single quantity.

The general form of a matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & \ddots & & & a_{2n} \\ a_{31} & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & \dots & \dots & \dots & a_{mn} \end{bmatrix}$$

The matrix order(or size) is represented as Rows by Columns (or) Rows $\times$ Columns. For example, the matrix $\begin{bmatrix} 5 & 4 & 3 \\ 10 & 7 & 2 \end{bmatrix}$ is a matrix of size 2×3.

Definition 1 Square matrix: *A square matrix is a matrix which has an equal number of rows and columns. The matrix size will be of the form $m \times n$ where $m, n \in \mathbb{N}$, where m, n represents the number of rows and columns, respectively. For example :-* $\begin{bmatrix} 4 & 6 \\ 9 & 20 \end{bmatrix}$ *is a matrix of size* 2×2.

All matrices are rectangular matrices. But the square matrix is a special type of rectangular matrix.

4.1 Identity matrix (I)

An identity matrix is a special type of square matrix. *An identity matrix is a square matrix in which all the elements of the principal diagonal are ones and all other elements are zeros.*

The *principal diagonal* is always the diagonal from the top left corner to the bottom right one.

Few examples are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

There only exists one identity for every order of square matrix.

4.2 Elementary Row Transformation

Elementary row transformation (ERT) are *operations done on the rows and columns of matrices to change their shape so that the computations become easier.* It is also used to discover the inverse of a matrix, the determinants of a matrix, and to solve a system of linear equations. There are 3 rules for ERT:-

1. $R_i \longleftrightarrow R_j$
   - Any 2 rows say i and j can be interchanged.
   - performing $R_2 \longleftrightarrow R_3$ on the below matrix we get,

- $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$

2. $R_i = k \cdot R_i$
    - k can be any number $\neq 0$.
    - Each term in the row should be multiplied by k.
    - Performing the operation $R_1 = 2 \cdot R_1$,
    - $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
    - This is the only operation which only uses 1 row.

3. $R_i = R_i + k \cdot R_j$
    - k can be any number $\neq 0$.
    - Performing the operation $R_3 = R_3 + 4 \cdot R_1$,
    - $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 11 & 16 & 21 \end{bmatrix}$
    - Again here each term of Row i is to be added with the k $\times$ corresponding term(Same column) of Row j.

4.3 A simple sample problem using ERT

Question:- Transform $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$ into I by ERT.

The order for solving the problem is as follows :-

1. Start with the first column.
2. Make the 1^{st} term 1 using ERT.
3. Make all other Terms in that column 0.
4. Go to the column and make the 2^{nd} term 1 using ERT.
5. Make all other terms in the 2^{nd} column 0.
6. Continue till the square matrix becomes I.

$$Start \longrightarrow \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \; R_1 = \frac{1}{2} \cdot R_1 \longrightarrow \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \; R_2 = R_2 + (-5)R_1 \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix}$$

$$R_2 = \frac{-1}{7} \cdot R_2 \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \; R_1 = R_1 + (-2)R_1 \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.4 Gauss-Jordan Elimination method

Suppose two equations with two variables are given and the solutions are asked, it is pretty simple to solve. If 3 equations are given? then not so much. If 4 or 5,6 equations are given, then it becomes non-trivial to solve. There is another way to solve set of equations using matrices and ERT. Here's a sample question :-

Solve:-

$$-x + 2y = 3$$

$$3x + 4y = 11$$

1. The coefficients of x and y, and the solutions on the RHS should be written as a matrix
   - $\begin{bmatrix} -1 & 2 & 3 \\ 3 & 4 & 11 \end{bmatrix}$
   - The first column corresponds to the x variable, the second column the y variable and the third column as the constant term.
   - Similarly, the first and the second row corresponds to the I^{st} and the II^{nd} equation, respectively.
2. The part of the matrix, where the coefficients are situated have to be converted to an identity matrix using ERT.
3. If we change the coefficient matrix into $[I]$, then the RHS side will be the solution.
4. $$Start \longrightarrow \begin{bmatrix} -1 & 2 & 3 \\ 3 & 4 & 11 \end{bmatrix} R_1 = -1 \cdot R_1 \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ 3 & 4 & 11 \end{bmatrix} R_2 = R_2 + (-3)R_1$$
   $$\longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ 0 & 10 & 20 \end{bmatrix} R_2 = \frac{1}{10} \cdot R_2 \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$
   $$\longrightarrow R_1 = R_1 + (-2)R_1 \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
5. So the answer is $x = 1, y = 2$. It can be confirmed with substitution.

This is how a set of equations are solved using the Gauss-Jordan elimination method. The same can be used to solve a set of equations with many variables.

5 The Simplex method

The combined simplex-graphical method was relatively easy, but when there are many constraints, this method is easier to use and feed to computers. *The simplex method deals with converting the inequalities to equations.* Basically in the ABC Ltd., problem, we have

$$x_1 \leq 4 \longrightarrow x_1 + x_3 = 4$$

$$2x_2 \leq 12 \longrightarrow 2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 \leq 18 \longrightarrow 3x_1 + 2x_2 + x_5 = 18$$

where x_1, x_2 and x_3 are known as '*Slack variables*'. These also follow non-negativity ($x_1, x_2, x_3 \geq 0$). The actual form and the augmented form is shown in Figure. 4.

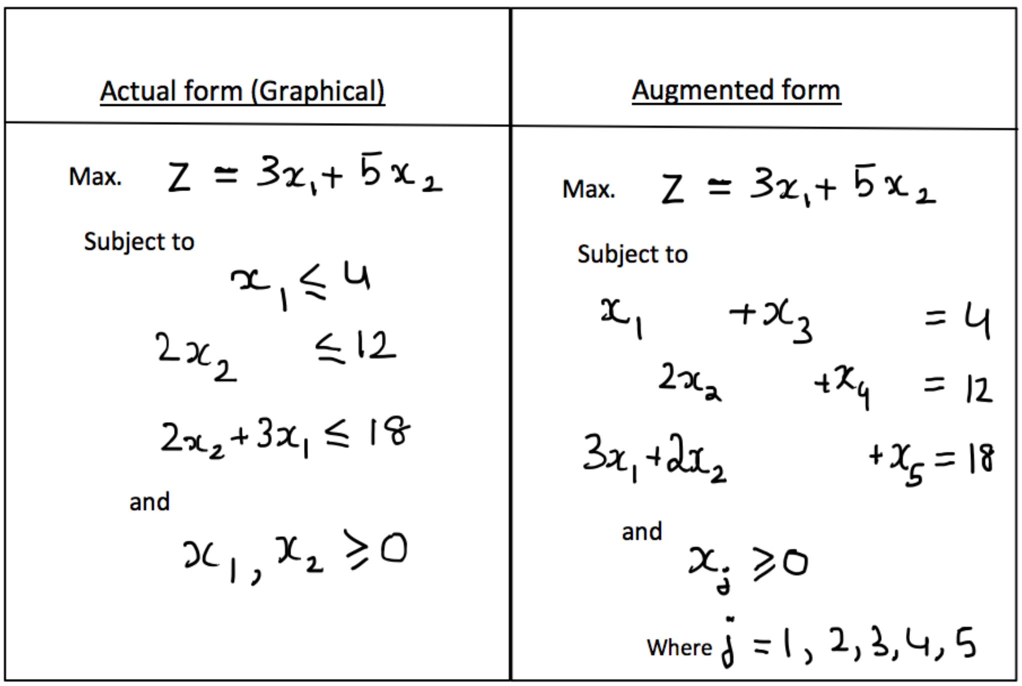

Figure 4: LPP Augmented form

For example:- augmenting $(x_1, x_2) = (3, 2)$, and by solving thee equations, we get $x_3 = 1, x_4 = 8$ and $x_5 = 5$

$$\therefore (3, 2) \longrightarrow (3, 2, 1, 8, 5) \longrightarrow feasible$$

$$(4, 6) \longrightarrow (4, 6, 0, 0, -6) \longrightarrow non-feasible$$

Degrees of freedom (D.O.F) = No. of variables - number of equations to solve the variables.

The variables can be divided into two categories, viz., basic and non-basic. The number of non-basic variables will be equal to the D.O.F and its value is always equal to 0. In this example, we have 2 non-basic variables, x_1 and x_2. The other three variables will be considered basic. When the production of a product is needed, the non-basic variables are moved to the basic variable and one of the basic variables shifts to the non-basic side(basically becomes 0)

5.1 The equations

Let's jot down the equations once more.

$$\begin{aligned} &(0) \quad z - 3x_1 - 5x_2 = 0 \\ &(1) \quad x_1 + x_3 = 4 \\ &(2) \quad 2x_2 + x_4 = 12 \\ &(3) \quad 3x_1 + 2x_2 + x_5 = 18 \end{aligned}$$

5.2 Initialisation

The process is started with initialisation, i.e. taking the worst possible case, $x_1, x_2 = 0, 0$.

Basic Variable	Equation no.	Coefficients of						RHS	Ratio
		z	x_1	x_2	x_3	x_4	x_5		
	0	1	-3	-5	0	0	0	0	
x_3	1	0	1	0	1	0	0	4	
x_4	2	0	0	2	0	1	0	12	12/2 = 6
x_5	3	0	3	5	0	0	1	18	18/2 = 9

Pivot column

Key element (Intersection of pivots)

Pivot row

1. Constructing a table as shown above.

2. Decide which non-basic variable should move to Basic (Rate of improvement of profit).

3. decide which basic variable is going to become non-basic.

The ratio in the table is defined as $\dfrac{\text{RHS}}{\text{coeff from Pivot column}}$. The coefficients taken should only be positive values. *In the row in which the ratio has the minimum value, that row's basic variables will become non-basic.* That row becomes the Pivot row. The column whose variable is to be moved out, becomes the Pivot column.

The next step:-

1. Make key element = 1 by ERT

- $R_{2_{new}} \longrightarrow \dfrac{R_2}{2}$

2. Make other elements from pivot column 0 by ERT.

    - $R_0 = R_0 + 5R_{2_{new}}$
    - $R_3 = R_3 - 2R_{2_{new}}$

After doing this we will get the next table where x_2 will be a basic variable and x_4 will be a non-basic one.

5.3 The next step

The next table is constructed and the new pivot column and row are chosen, again the pivot column undergoes ERT to become a column of an identity matrix, with the key element $\longrightarrow$ 1 and all other elements $\longrightarrow$ 0.

The graph after the first step's ERT changes will look like this.

Basic Variable	Equation no.	Coefficients of						RHS	Ratio
		z	x_1	x_2	x_3	x_4	x_5		
	0	1	-3	0	0	5/2	0	30	
x_3	1	0	1	0	1	0	0	4	4/1 = 4
x_2	2	0	0	1	0	1/2	0	6	
x_5	3	0	3	0	0	-1	1	6	6/3 = 2

Next we choose x_1 as the variable, moving out to the basic variables. Following the steps set earlier, we get:-

1. $R_{3_{new}} \longrightarrow \dfrac{R_3}{3}$
2. $R_0 = R_0 + 3R_{3_{new}}$
3. $R_1 = R_0 - R_{3_{new}}$

5.4 The final step

After doing the necessary ERTs it will look like this:-

Basic Variable	Equation no.	Coefficients of						RHS	Ratio
		z	x_1	x_2	x_3	x_4	x_5		
	0	1	0	0	0	3/2	1	36	
x_3	1	0	0	0	1	1/3	-1/3	2	
$\boxed{x_2}$	2	0	0	1	0	1/2	0	$\boxed{6}$	
$\boxed{x_1}$	3	0	1	0	0	-1/3	1/3	$\boxed{2}$	

Since there are no negative values in the $(0)^{\text{th}}$ row, we have reached the optimal solution. Looking at the RHS values of the table, we have,

$$z = 36 \longrightarrow (x_1, x_2) = (2, 6)$$

which is the answer that was obtained by the previous methods. The solution can be written as :-

$$(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$$

6 Conclusion

The final answer obtained from the simplex method is congruent to the answers obtained by other methods too. This method may seem difficult, but when a lot of constraints are placed, this becomes much easier and methodical than the other methods. The simplex-table method is the most widely used method for linear programming (at least the basics). This is all for LPP.☺

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☺THANK YOU☺