Student Corner

Rational 'Twin' Isosceles Triangles

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Problem. Can there exist two non-congruent isosceles triangles with same perimeter and same area? If yes, how can you find them? How many solutions exist? Can you find a complete parametrisation for such triangles?

This problem is from the *Ganitasarasangraha*, an ancient mathematical text written by the Indian mathematician Mahavira in 850 CE. It is one of the 'Paishachika' problems, i.e., Devilishly Hard Problems!

We address a modified version of the same problem. Specifically, we provide a general formula to generate all pairs of rational-sided isosceles triangles that share the same perimeter and the same rational area. By appropriate scaling, we can also make all these quantities integers.

Consider two isosceles triangles $\triangle ABC$ and $\triangle XYZ$ with rational sides:

$$(AB, AC, BC) = (a, a, b), \qquad (XY, XZ, YZ) = (u, u, v)$$

The perimeter of $\triangle ABC = 2a + b$, and the perimeter of $\triangle XYZ = 2u + v$. As the triangles have equal perimeter, we have $2a + b = 2u + v$.

We now compute the areas of the triangles using Heron's formula:

$$\text{Area of } \triangle ABC = \sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a+b}{2} - a\right)\left(\frac{2a+b}{2} - a\right)\left(\frac{2a+b}{2} - b\right)}$$

$$= \frac{b}{2}\sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right)}.$$

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Similarly, we can conclude that

$$\text{Area of } \triangle XYZ = \frac{v}{2}\sqrt{\left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right)}.$$

Now, since $\triangle ABC$ and $\triangle XYZ$ have equal area, we have:

$$\frac{b}{2}\sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right)} = \frac{v}{2}\sqrt{\left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right)},$$

$$\therefore\ b^2(2a+b)(2a-b) = v^2(2u+v)(2u-v). \tag{1}$$

In equation (1), we can note that $(2a+b)$ and $(2u+v)$ are the same because they are the perimeters of $\triangle ABC$ and $\triangle XYZ$ respectively. Therefore we have:

$$b^2(2a-b) = v^2(2u-v). \tag{2}$$

Let P denote the (equal) perimeters of $\triangle ABC$ and $\triangle XYZ$, i.e., $2a+b = P = 2u+v$. Write $2a-b$ as $P-2b$ and $2u-v$ as $P-2v$.

Substituting in equation 2 we have,

$$b^2(P-2b) = v^2(P-2v). \tag{3}$$

Evaluating and rearranging equation (3), we have:

$$P = \frac{2(b^3-v^3)}{b^2-v^2} = \frac{2(b-v)(b^2+bv+v^2)}{(b+v)(b-v)} = \frac{2(b^2+bv+v^2)}{b+v} \tag{4}$$

Now, we can find the equal sides of $\triangle ABC$ and $\triangle XYZ$ as follows:

$$a = \frac{P-b}{2} = \frac{1}{2}\left(\frac{2(b^2+bv+v^2)}{b+v} - b\right)$$

$$\therefore\ a = \frac{b^2+bv+2v^2}{2(b+v)}.$$

Similarly,

$$u = \frac{2b^2+bv+v^2}{2(b+v)}.$$

Let A denote the (equal) area of the two triangles. From Heron's Formula, we know that:

$$16A^2 = b^2(2a+b)(2a-b) = v^2(2u+v)(2u-v).$$

Let

$$g^2 = \left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right), \qquad h^2 = \left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right),$$

where g and h are rational numbers. Substituting in terms of P yields:

$$g^2 = \left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right) = \left(\frac{P}{2}\right)\left(\frac{P}{2}-b\right),$$
$$g^2 = \frac{1}{4}\left(\frac{2(b^2+bv+v^2)}{b+v}\right)\left(\frac{2(b^2+bv+v^2)}{b+v}-2b\right) = \frac{(b^2+bv+v^2)v^2}{(b+v)^2}.$$

Now, since

$$g^2 = \frac{(b^2+bv+v^2)v^2}{(b+v)^2},$$

it must be that b^2+bv+v^2 is a rational square.

We reach the same conclusion if we work with h.

We proceed to find the rational parametrization. For now, we take $b^2+bv+v^2=1$. Later we scale the sides of both triangles to allow this value to be any rational square.

For $b^2+bv+v^2=1$, an obvious solution is $(b,v)=(1,0)$. Now we simply draw a secant via this trivial solution, to the coordinates (b,v), the non-trivial solution. Let this line be $v=-k(b-1)$.

By substitution we conclude that $b^2-k(b-1)b+k^2(b-1)^2=1$. One solution to this quadratic is $b=1$ (from our trivial solution). We can use Vieta's formulas to find our other non-trivial root.

$$(k^2-k+1)b^2-(2k^2-k)b+(k^2-1)=0 \implies b=\frac{k^2-1}{k^2-k+1}$$

Now, we can evaluate v using the value of b:

$$v=-k(b-1)=-k\left(\frac{k^2-1}{k^2-k+1}-1\right)=\frac{2k-k^2}{k^2-k+1},$$
$$(b,v)=\left(\frac{k^2-1}{k^2-k+1},\frac{2k-k^2}{k^2-k+1}\right).$$

Thus we have found (b,v).

Note that (b,v) must be positive as they are the lengths of the sides of the triangle. The denominator k^2-k+1 is always positive, as it has negative discriminant. Hence the numerators must be positive.

$$k^2-1>0 \implies k\in(-\infty,-1)\cup(1,\infty), \tag{5}$$
$$2k^2-k>0 \implies k\in(0,2). \tag{6}$$

From equations (5) and (6) we can conclude that $1<k<2$, that is, k is a positive rational number between 1 and 2. Now we again evaluate the perimeter P.

$$\begin{aligned} P &= \frac{2(b^2+bv+v^2)}{b+v} \\ &= \frac{2}{\left(\frac{k^2-1}{k^2-k+1}\right)+\left(\frac{2k-k^2}{k^2-k+1}\right)} \\ &= \frac{2(k^2-k+1)}{2k-1}. \end{aligned} \tag{7}$$

Now, from equation (7), we know what the perimeter is. We can evaluate a, u again using the perimeter to finally obtain the quadruple:

$$a = \frac{P-b}{2} = \frac{1}{2}\left(\frac{2(k^2-k+1)}{2k-1} - \frac{k^2-1}{k^2-k+1}\right) = \frac{2k^4-6k^3+7k^2-2k+1}{2(2k-1)(k^2-k+1)} \tag{8}$$

Similarly, we have:

$$u = \frac{P-v}{2} = \frac{1}{2}\left(\frac{2(k^2-k+1)}{2k-1} - \frac{2k-k^2}{k^2-k+1}\right) = \frac{2k^4-2k^3+k^2-2k+2}{2(2k-1)(k^2-k+1)} \tag{9}$$

Thus we can conclude that:

$$\begin{aligned}(a,u) &= \left(a\left(\frac{2k^4-6k^3+7k^2-2k+1}{2(2k-1)(k^2-k+1)}\right),\ a\left(\frac{2k^4-2k^3+k^2-2k+2}{2(2k-1)(k^2-k+1)}\right)\right),\\ (b,v) &= \left(a\left(\frac{k^2-1}{k^2-k+1}\right),\ a\left(\frac{2k-k^2}{k^2-k+1}\right)\right),\end{aligned}$$

where $k \in (1,2)$ is rational and a is a positive rational (scaling factor).

Corollary 1. *There exist infinitely many pairs of non-congruent isosceles triangles having rational (or integer) sides, and both equal and rational (or integer) area and perimeter.*

We also make another claim regarding the number of triangles simultaneously having the same area and perimeter:

Corollary 2. *Three pairwise non-congruent isosceles triangles with rational sides cannot have equal and rational perimeter and area.*

Proof. Assume the contrary. Let the sides of the three isosceles triangles be (a,a,b), (u,u,v) and (c,c,d). From our previous arguments:

$$\begin{aligned} P = \frac{2(b^2+bv+v^2)}{b+v} = \frac{2(b^2+bd+d^2)}{b+d} &\implies \frac{v^2}{b+v} = \frac{d^2}{b+d},\\ v^2(b+d) = d^2(b+v) &\implies b(v^2-d^2)+vd(v-d)=0,\\ b(v+d)+vd=0 &\implies b = -\frac{vd}{v+d}.\end{aligned}$$

This is clearly impossible as b is the side of a triangle and hence cannot be negative. This proves the corollary by contradiction.

Since the area and the sides of the triangles are all rational, all of the altitudes of the triangles are rational as well. The median AD in $\triangle ABC$ as well as the median XU in $\triangle XYZ$ are also rational, since they are altitudes.

An interesting follow-up problem would be to ask whether it is also possible to have the medians $BE = CF = m_x$ and $YV = ZW = m_y$ to be rational as well. We leave this for the reader to explore!

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