Student Corner

Galaxy of Unit Fractions with Tom and Jerry

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In Prithwijit De's article in the July 2021 issue of *At Right Angles* (page 59), Jerry had asked Tom the following question:

Problem 1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all triples $a, b, c \in S$ with $a < b$, $c \neq a$, $c \neq b$, such that the following is true for all integers $n \geq 0$:

$$\frac{a}{b} = \frac{\overbrace{ccc\ldots cc}^{n}a}{b\underbrace{ccc\ldots cc}_{n}}. \tag{1}$$

(Here, $\overbrace{ccc\ldots cc}^{n}a$ denotes the $(n+1)$-digit number whose first n digits are c and last digit is a. Similarly, $b\underbrace{ccc\ldots cc}_{n}$ denotes the $(n+1)$-digit number whose first digit is b and last n digits are c.)

Tom is still looking for an answer!

I had written to Tom, stating that Problem 1 has no solutions. Tom, in response, decided to pose a problem of his own and wrote the following on the board.

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Challenge 1.

$$\frac{1}{3} = 0.\overline{01},$$
$$\frac{1}{5} = 0.\overline{0011},$$
$$\frac{1}{9} = 0.\overline{000111}, \quad \ldots$$

Can you identify the pattern and verify if this relationship is true for all integers $n \geq 0$?

Here the line over the 'decimal' indicates the recurring pattern: $0.\overline{01}$ means 0.01 01 01 01 ...; $0.\overline{0011}$ means 0.0011 0011 0011 0011 ...

Jerry could easily sort out the pattern on the LHS as $\frac{1}{2^n+1}$ but could not get things right on the RHS. But he had trained under Tom and knew all about his tactics. He was intensely observing the pattern to find the missing link.

Voila! He got it! The missing link is the base system. Tom had intentionally avoided mentioning that he was using the binary base system. (To understand non-decimal number bases better, the reader could refer to the 'Pullout' of the March 2022 issue of *At Right Angles*.) Quickly coming up with the proof was now a piece of cheesecake for Jerry.

Proof. We consider a typical positive integer, say $n = 3$. We wish to prove that

$$\frac{1}{2^3+1} = 0.\overline{000111} \quad \text{(in base 2)}. \tag{2}$$

Let $x = 0.\overline{000111}$ in base 2. Then we have $8x = 0.\overline{111000}$ (remember that in base 2, multiplication by 8 results in moving the 'decimal point' 3 places to the right, just like what multiplication by 1000 does in base 10). Hence by addition we get

$$\begin{aligned} 9x &= 0.\overline{000111} + 0.\overline{111000} \quad \text{(in base 2)} \\ &= 0.\overline{111111} = 0.\overline{1} \quad \text{(in base 2)} \\ &= 1. \qquad \text{(Comment. This is the base 2 equivalent of the base 10 relation } 0.99999\ldots = 1.) \end{aligned}$$

Therefore $x = \frac{1}{9}$. This shows that (2) is true.

Though we have written the solution only for the case $n = 3$, this approach clearly works for all integers $n \geq 0$. ☐

Jerry did not stop with this finding but did some more research and extended this result to other bases:

$$\frac{1}{2^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{111\ldots11}_{n}} \quad \text{(in base 2)}, \tag{3}$$

$$\frac{1}{3^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{222\ldots22}_{n}} \quad \text{(in base 3)}, \tag{4}$$

$$\frac{1}{4^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{333\ldots33}_{n}} \quad \text{(in base 4)}, \quad \ldots \tag{5}$$

and:

$$\frac{1}{8^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{777\ldots77}_{n}} \quad \text{(in base 8)}, \tag{6}$$

$$\frac{1}{10^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{999\ldots99}_{n}} \quad \text{(in base 10)}, \tag{7}$$

$$\frac{1}{16^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{FFF\ldots FF}_{n}} \quad \text{(in base 16)}, \quad \ldots \tag{8}$$

and so on.

Tom was unable to trap Jerry now and tried to challenge Jerry with a new question:

Challenge 2.

What is the corresponding relationship when $+1$ is replaced by -1 in the above relations (for all integers $n \geq 0$)?

Jerry in a split-second gave the following answer:

$$\frac{1}{2^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base 2)}, \tag{9}$$

$$\frac{1}{10^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base 10)}, \tag{10}$$

and in general:

$$\frac{1}{b^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base } b\text{)}, \tag{11}$$

for any base $b > 1$.

Proof. The proof was very simple for Jerry. We consider a typical positive integer, say $n = 3$. We wish to prove that

$$\frac{1}{2^3-1} = 0.\overline{001} \quad \text{(in base 2)}. \tag{12}$$

Let $y = 0.\overline{001}$ in base 2. Then we have $2y = 0.\overline{010}$ and $4y = 0.\overline{100}$ (remember that in base 2, multiplication by 2 results in moving the 'decimal point' 1 place to the right, and multiplication by 4 results in moving the 'decimal point' 2 places to the right). Hence by addition we get

$$y + 2y + 4y = 0.\overline{111} = 0.\overline{1},$$

i.e., $7y = 1$. Hence $y = \frac{1}{7}$, so (12) is true. □

Jerry then came up with the following generic table where b is the base:

b	$\frac{1}{b^n+1}$	$\frac{1}{b^n}$	$\frac{1}{b^n-1}$
2	$0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{111\ldots11}_{n}}$	$0.\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}}$
3	$0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{222\ldots22}_{n}}$	$0.\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}}$
10	$0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{999\ldots99}_{n}}$	$0.\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}}$

And in general, for base $b > 1$:

$$\frac{1}{b^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{(b-1)(b-1)(b-1)\ldots(b-1)(b-1)}_{n}},$$

$$\frac{1}{b^n} = 0.\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1},$$

$$\frac{1}{b^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\,\underbrace{1}_{1}}.$$

Now it was Jerry's turn to revert. He posed the following challenge to Tom:

Challenge 3.

Can the relationship in the table be extended to non-unit fractions in base k ($k \geq 2$), i.e., to fractions whose numerator is not 1?

Tom is now looking to the readers to provide an answer.

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