Summing V
a correction

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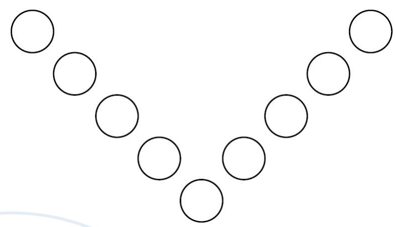
In the March 2020 issue of *At Right Angles*, as part of the 'Low Floor High Ceiling Tasks' series [1], the following problem was studied.

In how many ways can nine given numbers in arithmetic progression be arranged in a V shape such that the sums of the numbers on both the arms of the V are equal?



In our exploration of the problem, we noticed that the solution given in the article had an error. Here we provide a corrected count for the number.

Without loss of generality, we assume the 9 numbers in arithmetic progression to be the integers 1, 2, . . . , 9. We have:

$$1+2+\cdots+9=\frac{9\cdot 10}{2}=45.$$

If x is the number at the bottom of the V, then the sum of the numbers in each arm (excluding the centre) must be $(45 - x)/2$. Clearly, x must be odd. So $x \in \{1,3,5,7,9\}$.

To start with, we ignore the actions of rearranging the numbers in the arms and mirroring the arms. To account for this, at the end we multiply by $2 \cdot (4!)^2$.

The cases when 1 and 9 are at the centre may be matched 1 – 1 with each other, by replacing each number k by $10 - k$, uniformly through the V. These two cases must therefore have the same number of possibilities. The same applies to 3 and 7. So we only need to focus on the cases when the central number is 5, 7 or 9.

When the central number is 9, the sum in each arm is $(45 - 9)/2 = 18$. Consider the pairs {1, 8}, {2, 7}, {3, 6} and {4, 5}. Each pair has the same sum, 9, which is half of 18. It follows that if

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even one of these pairs stays intact (i.e., has both numbers on the same arm of the V), then all the pairs must stay intact. By fixing {1,8} on one arm, we have 3 choices for the other pair which must accompany it, thus making for 3 possibilities. Once this is done, there are no more choices possible. Thus there are 3 possibilities in which all the pairs stay intact.

Next, consider the case where 1 and 8 lie on different arms. The three other numbers on the same arm as 8 must add to 10. Since 3 + 4 + 5 > 10, the smallest number on that arm must be 2. The two remaining numbers on that arm must now add to 8. There is just one possible way of achieving this: 3 + 5. It follows that the numbers on the same arm as 8 are {2, 3, 5, 8}, which means that the numbers on the other arm are {1, 4, 6, 7}. Observe that with 1 and 8 on different arms, there is just one way of assigning the remaining numbers to the two arms.

So, with 9 at the centre of the V, there are a total of 3 + 1 = 4 possibilities. So there are also 4 possibilities with 1 at the centre of the V.

Next, consider the case when the central number is 7. The sum of the numbers on each arm is (45 – 7)/2 = 19. As the sum is odd, each arm must have an odd number of odd numbers. As there are four odd numbers available (namely, {1, 3, 5, 9}), one arm must have three odd numbers and the other arm must have one odd number. There are 4 ways of choosing 3 odd numbers from the collection {1, 3, 5, 9}. The choice 1, 3, 5 forces the fourth number on that arm to be 10, which is not admissible; so this choice is not available. The other three choices all lead to valid solutions:

Choice of 3 numbers	Solution for the V
{1,3,9}	{1,3,9,6} | {7} | {2,4,5,8}
{1,5,9}	{1,5,9,4} | {7} | {2,3,6,8}
{3,5,9}	{3,5,9,2} | {7} | {1,4,6,8}

This gives 3 possibilities each for 3 and 7 at the centre.

Finally, we consider the case when 5 is at the centre. The sum of the numbers on each arm is (45 – 5)/2 = 20. As the sum is even, each arm must have an even number of odd numbers. The odd numbers available are {1, 3, 7, 9}. We could have all the odd numbers on the same arm; this leads to a solution since 1 + 3 + 7 + 9 = 20. Else, we must have two odd numbers on each arm. If {1, 3} are on the same arm, then the other two numbers can only be {7, 9}, which leads to the solution already listed; so we do not consider this possibility. There are two other possibilities, and both lead to valid solutions. If {1, 7} are on the same arm, then the other two numbers can only be {4, 8}; the numbers on the other arm are then {3, 9, 2, 6}. Finally, if {1, 9} are on the same arm, the other two numbers being even, then the other two numbers can be {4, 6} or {2, 8}. This possibility thus leads to two valid solutions. It follows that with 5 at the centre, there are a total of 1 + 1 + 2 = 4 possibilities.

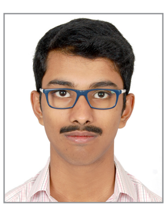
Our analysis thus yields a total of 4 + 3 + 4 + 3 + 4 = 18 possibilities.

Taking rearrangements into account, it follows that the number of ways of filling the V according to the required conditions is

$$2 \cdot (4!)^2 \cdot 18 = 20736.$$

References

[1] Math Space, "Summing V" from https://azimpremjiuniversity.edu.in/SitePages/resources-ara-vol-9-no-6-march-2020-summing-V.aspx



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