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Two Striking Number Patterns

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Sums of squares of the natural numbers from the Pascal triangle

The array below shows the first 12 rows of the Pascal triangle. (*Editor's note*. The triangle has been typeset in a left justified manner, different from the usual depiction.)

1											
1	1										
1	2	1									
1	3	3	**1**								
1	4	6	**4**	1							
1	5	10	**10**	5	1						
1	6	15	**20**	15	6	1					
1	7	21	**35**	35	21	7	1				
1	8	28	**56**	70	56	28	8	1			
1	9	36	**84**	126	126	84	36	9	1		
1	10	45	**120**	210	252	210	120	45	10	1	
1	11	55	**165**	330	462	462	330	165	55	11	1

Here are the entries of the *fourth column* of this triangular array (with a 0 included in the front), written in row form:

0, 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, ...

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We add *pairs of consecutive members* of this sequence and get these numbers:

$$1,\ 5,\ 14,\ 30,\ 55,\ 91,\ 140,\ 204,\ 285,\ 385,\ 506,\ 650,\ 819,\ 1015,\ 1240,\ \ldots.$$

We have obtained the sums of squares of the consecutive natural numbers:

$$\begin{aligned}
1 &= 1^2,\\
5 &= 1^2+2^2,\\
14 &= 1^2+2^2+3^2,\\
30 &= 1^2+2^2+3^2+4^2,\\
55 &= 1^2+2^2+3^2+4^2+5^2,\\
91 &= 1^2+2^2+3^2+4^2+5^2+6^2,\\
140 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2,\\
204 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2,\\
285 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2,\\
385 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2,
\end{aligned}$$

and so on.

Relationship between the cubes and the triangular numbers

Starting with the sequence of cubes (1, 8, 27, 64, 125, . . .), we compute the differences between pairs of consecutive cubes, then the differences between consecutive numbers of that sequence, and so on. Here is what we get:

1		8		27		64		125		216		...
	7		19		37		61		91		...	
		12		18		24		30		...		
			6		6		6		...			

We see that in the third row (12, 18, 24, . . .), the numbers are consecutive multiples of 6, starting with 12. This means that the numbers in the second row (7, 19, 37, 61, . . .) all leave the same remainder under division by 6. That common remainder is 1. Subtracting 1 from all the numbers in this row, we get these numbers which are all multiples of 6:

$$6,\ 18,\ 36,\ 60,\ 90,\ \ldots.$$

Dividing by 6, we get these numbers:

$$1,\ 3,\ 6,\ 10,\ 15,\ \ldots.$$

These are the triangular numbers.

Therefore, if T_n denotes the n-th triangular number, we have discovered the following relationship:

$$\boxed{6T_n + 1 = (n+1)^3 - n^3.}$$

Editor's note: Sums of squares of natural numbers from the Pascal triangle

We provide an explanation behind the observed relationship. Note that the numbers $1, 4, 10, 20, 35, 56, \ldots$ are the partial sums of the sequence of triangular numbers $1, 3, 6, 10, 15, \ldots$:

$$\begin{aligned} 1 &= 1, \\ 4 &= 1 + 3, \\ 10 &= 1 + 3 + 6, \\ 20 &= 1 + 3 + 6 + 10, \\ 35 &= 1 + 3 + 6 + 10 + 15, \quad \text{and so on.} \end{aligned}$$

So if a_n denotes the n-th number of the sequence $1, 4, 10, 20, 35, \ldots$, then we have

$$a_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

By adding consecutive members of the a-sequence, we obtain

$$a_{n-1} + a_n = (T_1 + T_2 + T_3 + \cdots + T_{n-1}) + (T_1 + T_2 + T_3 + \cdots + T_{n-1} + T_n).$$

This relationship may be written as follows:

$$a_{n-1} + a_n = T_1 + (T_1 + T_2) + (T_2 + T_3) + \cdots + (T_{n-1} + T_n).$$

The following relationships are well-known:

$$T_1 = 1^2, \quad T_1 + T_2 = 2^2, \quad T_2 + T_3 = 3^2, \quad \ldots, \quad T_{n-1} + T_n = n^2.$$

Hence we have the identity:

$$a_{n-1} + a_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

This explains the observation made by Adithya.

Box 1

Editor's note: Relationship between cubes and triangular numbers

The triangular numbers are generated by the following formula:

$$T_n = \frac{n(n+1)}{2},$$

so $6T_n + 1 = 3n(n+1) + 1$. This means that we must check whether the following is an identity:

$$3n(n+1) + 1 = (n+1)^3 - n^3.$$

It is an easy exercise to check that this is an identity; both sides simplify to the expression $3n^2 + 3n + 1$. Hence proved.

Box 2

ADITHYA RAJESH is a 8-year old boy currently studying in Class 3 in PSBB KK Nagar, Chennai. He has been passionate about numbers from a very young age and gets fascinated by the beauty of number patterns. At present, his interest lies in Number Theory and Permutations and Combinations. *At Right Angles* has enabled him to leap one step forward in his thinking and he treasures this magazine. He also enjoys playing Carnatic music on the keyboard. He loves the mathematical aspects of the ragas and thalas of Carnatic music.